

DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

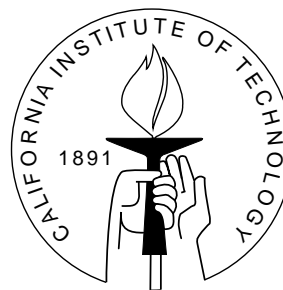
PASADENA, CALIFORNIA 91125

A DYNAMIC SCHOOL CHOICE MODEL

Juan Sebastián Pereyra

El Colegio de México

California Institute of Technology



SOCIAL SCIENCE WORKING PAPER 1347

May 2011

A Dynamic School Choice Model

Juan Sebastián Pereyra

Abstract

This paper considers a real-life assignment problem faced by the Mexican Ministry of Public Education. Inspired by this situation, we introduce a dynamic school choice problem that consists in assigning positions to overlapping generations of teachers. From one period to another, agents are allowed either to retain their current position or to choose a preferred one. In this framework, a solution concept that conciliates the fairness criteria with the individual rationality condition is introduced. It is then proved that a fair matching always exists and that it can be reached by a modified version of the deferred acceptance algorithm of Gale and Shapley. We also show that the mechanism is dynamic strategy-proof, and respects improvements whenever the set of orders is lexicographic by tenure.

JEL classification numbers: C71; C78; D71; D78; I28

Key words: School choice; Overlapping agents; Dynamic matching; Deferred acceptance algorithm

A Dynamic School Choice Model*

Juan Sebastián Pereyra

1 Introduction

Since David Gale and Lloyd Shapley published their famous paper "College admissions and the stability of marriage" ([9]), many authors have studied assignment problems in different contexts. Therefore, there is an extensive literature on allocation problems, which, primarily considers static models. In contrast, there are many real-life applications where an assignment is made in a dynamic context. Some examples are on-campus housing for college students, in which freshmen apply to move in and graduating seniors leave (Kurino [10]), kidney exchange of patients, in which each agent arrives with an object to trade (Ünver [14]), and firms with workers whose entry and exit lead to a reassignment of fixed resources (Bloch and Cantala [4]). In this paper, we study a dynamic version of the well-known school choice model. Specifically, our model assigns school positions to overlapping generations of teachers. In each period, the central authority must assign positions to teachers, taking into account each school's priority ranking and the

*This paper constitutes the first chapter of my Ph.D dissertation at El Colegio de México. It is written under the supervision of David Cantala within the CONACYT project 62188. I am also grateful to Jordi Massó, Szilvia Papai, Federico Echenique, Christine Daley, Kaniska Dam, Alexander Elbittar, Rafael Treibich, Francis Bloch, Emerson Melo, SangMok Lee, Fuhito Kojima, Edwin van Gamen, Juan Gabriel Brida, Juliana Xavier, Andrés Sambarino and Rodrigo Velez for their comments and suggestions and the seminar participants at École Polytechnique, Economics Department at Universidad de la República de Uruguay and El Colegio de México for discussions. All errors are my responsibility.

previous matching. From one period to another, agents are allowed either to retain their current position or to choose a preferred one (if available). Hence, the central authority faces a dynamic allocation problem.

The original motivation for this paper is an assignment problem faced by the Mexican Ministry of Public Education. In May 2008 the Mexican Federal Government, through the Ministry of Public Education (SEP), signed an agreement with the National Education Workers Union (SNTE) called "The Alliance for the Quality of Education."¹ Part of the agreement was the creation of the National Contest for the Allocation of Teaching Positions, a mechanism to assign teachers to teaching positions. As a consequence of this agreement, teachers looking for a position in the public education system are required to sit an exam. According to each teacher's grade, the central authority ranks the teachers and then assigns each a teaching position. Specifically, under the mechanism used by the central authority, all open positions (that is, positions that are not already assigned) are offered to the first teacher in the ranking. Once this first ranked teacher chooses a school, remaining open positions are offered to the second teacher, and so on. Moreover, any teacher that had been previously assigned a position may choose her current position over the new positions that are offered to her. Thus, the central authority applies a variant of the serial dictatorship mechanism, which takes into account that some agents are initially assigned a position. In 2010, 145,983 teachers participated in the exam in order to obtain a position.

Cantala [5] shows that the mechanism has some major flaws (see Appendix A.1 for an illustrative example). In particular, a teacher can profit in a period after she enters the market by misrepresenting her preferences. This implies that the mechanism is not dynamic strategy-proof: it can be manipulated by teachers. Another flaw is that the mechanism does not respect improvements made by teachers (Balinski and Sönmez [3]), that is, a teacher may increase her order in one school's priority ranking, but be assigned to a worse position. In this paper, we study the described problem within a more general framework in order to cast some light on the resource allocation problem faced by SEP

¹More information can be obtained from <http://www.concursonacionalalianza.org>

and SNTE.

A central concept in matching theory is stability: a matching is stable if there is no unmatched teacher-school pair in which the teacher prefers another school to her assignment and there is a teacher with lower priority assigned to that school. In school choice models, this concept is usually referred to as the elimination of justified envy (Abdulkadiroglu and Sonmez [2]) and embodies a notion of fairness. In addition to elimination of justified envy, since we cannot assign a teacher to a less preferred school than the one where she is teaching, we have to address the individual rationality condition. We present a new fairness notion to accommodate these concepts.

In order to define a fair matching, we consider the claims that could exist in an individual matching. Usually it is said that a teacher has a claim over a school if there exists a school that she prefers over her assignment, and she has higher priority for it than one of the assigned teachers. Note that a matching eliminates the justified envy if and only if there is no claim in the matching. Moreover, we consider two kinds of claims. If the teacher in the preferred school was not assigned to it in the previous period, we say that it is a justified claim. On the contrary, if the teacher was assigned to the school in the previous period, the claim is considered inappropriate. Observe that the last type of claim is inappropriate due to the individual rationality restriction. Finally, our fairness concept is as follows. We say that a matching is fair if:

- it is individually rational, non-wasteful (whenever a teacher prefers a school to her own assignment, that school already has all its slots filled), and does not have justified claims; and

- if there are inappropriate claims, the following must hold: there is no other matching that satisfies the three previous properties and one inappropriate claim is *solved* without creating a new one.

It is worth noting that SEP did not propose an explicit fairness concept and, also, that the mechanism which is used by this central authority does not satisfy our definition of fairness.

In this context, we show that there exists a unique fair matching Pareto superior to any other fair matching. In order to find it, a modified version of the deferred acceptance algorithm of Gale and Shapley is introduced. Before applying the algorithm, we modify each school's priority ranking by moving teachers who had been assigned to the school in the previous period to the top of the school's priority ranking.² With these new orders we define the related market in which the deferred acceptance algorithm is applied.

A new dynamic version of strategy-proofness is introduced. The classic concept in static matching problems only makes reference to the benefit in one period. Our notion of strategy-proofness is dynamic in the sense that it involves not only the period when the teacher enters the market but also all the later periods while she is in the market. In our framework, teachers reveal their preferences in the period in which they enter the market. In the following periods, they cannot modify the announced preferences. We prove that if each school's priority ranking is lexicographic by tenure, that is, if teachers who were present in the previous period have priority over new teachers, then the proposed mechanism is dynamic strategy-proof. Finally, it is shown under the same condition that the mechanism also respects improvements made by teachers. Our concept of respecting improvements involves not only the period when the teacher improves her position in the ranking (like the classic notion), but also every following period.

As we mentioned, the literature on matching is mostly devoted to static matching problems (see, for example, the excellent surveys of Roth and Sotomayor [12] and Sönmez and Ünver [13]). Recently, some articles have presented assignment problems in dynamic contexts. Kurino [10] is closest to our model. The author introduces a model of house allocation with overlapping agents and analyzes the impact of orderings on Pareto efficiency and strategy-proofness. In this sense, it is shown that under time-invariant preferences, orders that favor existing tenants perform better, in terms of Pareto efficiency and strategy-proofness, than those that favor newcomers. The author also studies two dynamic mechanisms: a spot mechanism (with or without property right transfers) and a future mechanism. Nevertheless, there are two main differences that distinguish

²A similar modification can be also found in Compte and Jehiel [6].

our work from Kurino's [10]. In the first place, we consider a fairness concept. We are interested in fair matchings because each school's priority ranking should be taken into account. In the second place, our notion of strategy-proofness is defined in a dynamic context and, in this sense, it takes into account all periods when the teacher is in the market.

The rest of the paper is organized as follows. In Section 2, we introduce the ingredients of our model and the fairness concept. Section 3 is devoted to the existence of a solution to our problem. In the next section, the proposed mechanism is introduced. Sections 5 and 6 analyze dynamics problems that arise in the model: dynamic strategy-proofness and respecting improvements properties. In Section 7, we present the conclusions and directions for future research.

2 Preliminary definitions

2.1 The Model

We consider the allocation of teaching positions to overlapping generations of teachers. Time is discrete, starts at $t = 1$, and lasts forever. In each period, there is a set of schools denoted by S . Each school $s \in S$ has q_s positions, and in each period, some of them can already be assigned and the others are open. Additionally, we have the null school, denoted by s_0 , which will be used to assign no school to teachers; we suppose that s_0 is not scarce. Denote by I^t the set of teachers in period t . Note that I^t changes over time because in each period some teachers may exit the market while new teachers may enter. We assume that $|I^t| \leq \sum_{s \in S} q_s$ for all t .

Another ingredient of the model is a set of strict priority orders for all teachers, denoted by $>^t \equiv \{>_s^t\}_{s \in S}$, which includes one different order for each school. When teacher i has priority over j to choose a position in school s in period t , we write $i >_s^t j$. We suppose that the relative order of teachers for each school does not change over time,

that is, if $i >_s^t j$ at some t , then $i >_s^\tau j$ for all τ such that $i, j \in I^\tau$.³

Each agent $i \in I^t$ has a complete and transitive preference relation over $S \cup \{s_0\}$, denoted by \succsim_i , and \succ_i is the induced strict preference relation over the same set. We assume that no teacher prefers the null school (to be unmatched) to a real school (that is, $s \succ_i s_0$ for all s and i). Teachers reveal their preferences in the period in which they enter the market. In the following periods, they cannot modify the announced preferences. Let Λ_i be the set of strict preference relations of agent i . A preference profile at t is an element of the Cartesian product of the set of preferences of all teachers present at t : $\Lambda = \prod_{i \in I^t} \Lambda_i$; we denote by $\succ = (\succ_i)_{i \in I^t}$ a preference profile at t .⁴

2.2 Matchings

A **matching** at t is an assignment of teachers to schools such that every agent is assigned one school, and no school has more teachers assigned than slots, i.e., a function $\mu_t : I^t \rightarrow S \cup \{s_0\}$ such that $|\mu_t^{-1}(s)| \leq q_s$ for all $s \in S$. To indicate that agent i is matched to school s in period t , we write $\mu_t(i) = s$. Let \mathcal{M}_t be the set of all matchings in period t . A **submatching** is a matching with restricted domain, i.e., a function $\nu_t : J \subset I^t \rightarrow S \cup \{s_0\}$.

In the **initial period**, we have a set of teachers (denoted by $I_E^1 \subset I^1$), each of whom is initially assigned to a school.⁵ The initial assignment can be considered as a submatching in which each teacher in the set I_E^1 is matched to her school. Hence, we describe the initial submatching of period 1 as a function $\nu_1 : I_E^1 \rightarrow S$ such that $\nu_1(i) = s$ if and only if i is initially matched to school s . For any period $t \geq 2$, the initial submatching, denoted by ν_t , is defined by the matching of the previous period; that is, given the matching of

³As it is common in this type of model, we assume that each school's priority ranking is responsive (see Roth and Sotomayor [12] for more details).

⁴Although formal notation would be \succ_t , to simplify it we will not use the subindex t . Then with \succ we will refer to teacher's preferences in the period under study.

⁵The subscript E is motivated by the fact that teachers in this group play the role of what is known in the literature as existing tenants.

the previous period μ_{t-1} and sets S, I^t , we have $\nu_t = \mu_{t-1} \mid I_E^t$ with $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$.⁶ Clearly, it should be that $|\nu_t^{-1}(s)| \leq q_s$ for all s . Note that $I^t \setminus I_E^t$ is the set of teachers who do not hold positions and are competing to hold one.

Given a matching μ_{t-1} , the sets S, I^t , the number of positions in each school $\{q_s\}_s$, the set of strict orders $>^t = \{>_s^t\}_s$, and the preference profile at t \succ , **an overlapping teacher placement problem** is represented by the market $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$. Notice that a market M^t defines the initial submatching of period t , since $\nu_t = \mu_{t-1} \mid I_E^t$ and $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$ if $t \geq 2$ (when $t = 1$ we have $\mu_0 \equiv \nu_1$). A solution of an overlapping teacher placement problem is a matching.

A **mechanism** is a systematic procedure that assigns a matching for each problem; that is, a function φ such that $\varphi(\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle) \in \mathcal{M}_t$, for any problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$. We will often abbreviate notation by omitting most of the arguments and we will write $\varphi(I^t, \succ)$. We believe that this abuse does not confuse and it makes the notation more manageable.

An **economy** is defined by the set of schools S and its slots $\{q_s\}_s$, an initial submatching ν_1 , sequences of sets $\{I^t\}_t$, preference profiles $\{\succ\}_t = \{(\succ_i)_{i \in I^t}\}_t$, strict priority orders of all teachers for each school $\{>^t\}_t$ and finally, the mechanism, denoted by φ . Observe that in the context of our model, the mechanism is included in the economy because the matching in one period links this period with the one following. Specifically, the matching in one period determines the initial submatching for the next period. Therefore, the mechanism plays the role of a transition rule between periods. Finally, note that an economy defines the market in each period.

2.3 Fairness

The remainder of this section is devoted to the definition of fairness. We combine two classic concepts present in the literature. On the one hand, since we have existing tenants

⁶Here $\mu_{t-1} \mid I_E^t$ means the restriction of function μ_{t-1} to the set I_E^t .

in our model, we cannot assign a teacher to a less preferred school than the one where she is teaching. Therefore, a fair matching should satisfy the individual rationality condition, as defined in Abdulkadiroglu and Sönmez [1]. On the other hand, we must respect the strict priority order of all teachers for each school. Hence, a fair matching should eliminate justified envy, as defined by Abdulkadiroglu and Sönmez [2].

Consider any period t of our model. The information included in the market in that period is given by $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ^t, >^t \rangle$. Then, the initial submatching of period t is defined by $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$ and $\nu_t = \mu_{t-1} \mid I_E^t$. Our goal is to define a fair matching for the market M^t . With that purpose, we first define the classic concepts of individual rationality and non-wastefulness:

- A matching is individually rational if no teacher prefers the null school option or the school she was initially assigned to her newly assigned school.
- A matching is non-wasteful if whenever a teacher prefers a school to her own assignment, that school already has all its slots filled.

Next, we consider the claims that could exist after the matching. We say that a teacher has a claim over a school if she prefers that school over her own assignment and if a lower ranked teacher (in the priority order) has been assigned to that school. Moreover, as we have explained, two kinds of claims can occur. The formal definitions are the following.

Definition 1 *A matching μ_t is **individually rational** if:*

- i) $\mu_t(i) \succsim_i s_0$, for all $i \in I^t$,
- ii) $\mu_t(i) \succsim_i \nu_t(i)$, for all $i \in I_E^t$.

Definition 2 *A matching μ_t is **non-wasteful** whenever a teacher $i \in I^t$ exists and a school s , such that $s \succ_i \mu_t(i)$ then $|\mu_t^{-1}(s)| = q_s$.*

Definition 3 *Given a matching μ_t , teacher i has a **justified claim** over school s if:*

i) i prefers s to her assignment: $s \succ_i \mu_t(i)$, and

ii) there exists a teacher k assigned to s such that i has priority over teacher k in the school's ranking, and k was not assigned to s in the previous period; that is: $\exists k \in I^t$ such that $\mu_t(k) = s$, $i >_s^t k$, and $k \notin \nu_t^{-1}(s)$.

In the last definition, the criteria take into account that teacher i has justified envy of the assignment of teacher k . Also note that if an agent in the preferred school was assigned to it in the previous period, the claim is inappropriate or not justified, because the agent has the right to continue in that school.

We say that a matching **eliminates the justified claims** if there is no justified claim in the matching.

Definition 4 *Given a matching μ_t , teacher i has an ***inappropriate claim*** over school s if:*

i) i prefers s to her assignment: $s \succ_i \mu_t(i)$, and

ii) there exists a teacher k assigned to s , such that i has priority over teacher k in the school's ranking, and k was assigned to s in the previous period; that is: $\exists k \in I^t$ such that $\mu_t(k) = s$, $i >_s^t k$, and $k \in \nu_t^{-1}(s)$.

According to Definition 4, if the teacher in the preferred school was assigned to it in the previous period, then she has the right to continue in that school even if she has a lower ranking.

We say that a matching **eliminates the inappropriate claims** if there is no inappropriate claim in the matching.

Let $\Gamma(\mu_t)$ be the set of all inappropriate claims in matching μ_t , that is: $\Gamma(\mu_t) = \{(i, s) \in I^t \times S, \text{ such that } i \text{ has an inappropriate claim over } s \text{ in } \mu_t\}$.

Definition 5 Matching μ_t is **acceptable** if it:

- i) is individually rational,
- ii) is non-wasteful, and
- iii) eliminates the justified claims.

Let $\mathcal{C}_t \subset \mathcal{M}_t$ denote the set of all acceptable matchings.

Before presenting our concept of fairness, we explain our motivation with the following example.

Example 1 Consider the following problem with $S = \{s_1, s_2, s_3\}$, $q_1 = 2$, $q_2 = q_3 = 1$, $I^t = \{i, j, k, l\}$, $\nu_t = \{(i, s_1), (j, s_2)\}$, and the following preferences (from best to worst) and orders:

$$\begin{bmatrix} \succsim_i & \succsim_j & \succsim_k & \succsim_l \\ s_3 & s_2 & s_1 & s_2 \\ s_2 & s_1 & s_3 & s_1 \\ s_1 & s_3 & s_2 & s_3 \end{bmatrix} \quad \begin{bmatrix} >_1^t & >_2^t & >_3^t \\ l & k & k \\ k & i & j \\ i & l & i \\ j & j & l \end{bmatrix}$$

The following matchings are acceptable:

$$\mu_t^1 = \begin{pmatrix} i & j & k & l \\ s_1 & s_2 & s_3 & s_1 \end{pmatrix} \quad \text{and} \quad \mu_t^2 = \begin{pmatrix} i & j & k & l \\ s_3 & s_2 & s_1 & s_1 \end{pmatrix}.$$

Note that $\Gamma(\mu_t^2) = \{(l, s_2)\} \subsetneq \Gamma(\mu_t^1) = \{(i, s_2), (k, s_1), (l, s_2)\}$. Then, it is clear that we should not define matching μ_t^1 as fair because there is another acceptable matching that solves two inappropriate claims without creating a new one. ■

Our concept of fairness captures the idea illustrated in the last example: a matching is fair if it is acceptable and, if it has inappropriate claims, there is no other acceptable matching solving one of these claims without creating a new one.

Definition 6 *A matching μ_t is **fair**:*

- i) if it is acceptable,*
- ii) there is no acceptable matching μ'_t such that $\Gamma(\mu'_t) \subsetneq \Gamma(\mu_t)$.*

If there is an acceptable matching without inappropriate claims then, by the previous definition, it is fair. Also notice that the concept of fairness does not imply a utilitarian perspective. Indeed, we may have two fair matchings μ_t, μ'_t , even if $|\Gamma(\mu_t)| < |\Gamma(\mu'_t)|$.

It is easy to verify in the last example that there is no fair matching without inappropriate claims. Therefore, we cannot in general guarantee the existence of a fair matching with $\Gamma(\mu_t) = \phi$. We can wonder, however, if it is possible to guarantee the existence of a fair matching in any overlapping teacher placement problem. The next section is devoted to that question.

3 Existence

In order to prove the existence of a fair matching, we introduce the concept of related market. We want to apply the deferred acceptance (DA) algorithm of Gale-Shapley to obtain an individually rational matching. With that purpose, we modify each school's priority ranking. In each new priority ranking, we have two groups of teachers. The first group in the new ranking is the set of teachers who had been assigned to the school in the previous period, and the second is the remaining teachers. Within each group, the order is defined by the original ranking $>_s^t$. With these new orders, we define the related market in which the DA algorithm is applied. By Ergin [8] Proposition 1, we know that the outcome of the DA algorithm adapts to the order structure: there is no teacher such

that there is a school that she prefers over her assignment, and she has priority for it over one of the assigned teachers. Next, we prove that the DA outcome is an acceptable matching in the original market. Finally, since the set \mathcal{C}_t is finite and not empty, we choose one acceptable matching with the fewer number of claims; this is a fair matching.

Definition 7 Let $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ be an overlapping teacher placement problem. For each school $s \in S$ with priority ranking $>_s^t$, let's define the following order of all teachers, denoted by O_s^t , as $i, j \in I^t$ and $s \in S$, if:

1. $i, j \in \nu_t^{-1}(s)$ the order is defined by $>_s^t$, that is $i O_s^t j \Leftrightarrow i >_s^t j$,
2. $i \in \nu_t^{-1}(s)$ and $j \notin \nu_t^{-1}(s)$, then $i O_s^t j$, and
3. $i, j \in I^t \setminus \nu_t^{-1}(s)$ the order is defined by $>_s^t$, that is $i O_s^t j \Leftrightarrow i >_s^t j$.

Let $O^t = \{O_s^t\}_{s \in S}$ be the set of all such orders indexed by the school. Then, given a market $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, the **related market** is $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$.⁷

Given the market $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ and the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, we have all elements to apply the DA algorithm of Gale and Shapley [9] to the related market. The algorithm works as follows:

Step 1. Each teacher proposes to her top choice. Each school s rejects all but the best q_s teachers among those teachers who proposed to it. Those that remain are “tentatively” assigned one slot at school s .

In general,

Step k. Each teacher who is rejected in the last step proposes to her top choice among those schools that has not yet rejected her. Each school s rejects all but the best q_s teachers among those teachers who have just proposed and those who were tentatively assigned to it at the last step. Those who remain are “tentatively” assigned one slot at school s .

⁷The idea of the related market in which position-specific priorities are modified can be also found in Compte and Jehiel [6].

The algorithm terminates when no teacher proposal is rejected. Each teacher is assigned to her final tentative assignment.

When we apply the DA algorithm, since $|\nu_t^{-1}(s)| \leq q_s$, if $\mu_t(k) \neq \nu_t(k)$ for some $k \in \nu_t^{-1}(s)$, then $\mu_t(k) \succ_k \nu_t(k)$. That is, using orders O^t and applying the DA algorithm, we obtain an individually rational matching.

Following Ergin [8], we present the following definition.

Definition 8 *Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ and the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, we say that matching μ_t **violates the priority** of i for s , if there is a teacher h such that $\mu_t(h) = s$, $s \succ_i \mu_t(i)$ and $i O_s^t h$. The matching μ_t **adapts** to O^t if it does not violate any priorities.*

The relation between a matching that adapts to O^t and an acceptable matching is straightforward, as we prove in the next lemma.

Lemma 1 *Given an overlapping teacher placement problem $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ and the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, a matching is acceptable (relative to the market M^t) if and only if it adapts to O^t (regarding the related market) and it is non-wasteful.*

Proof. (\Rightarrow) An acceptable matching is, by definition, non-wasteful. Then, suppose that μ_t is acceptable but violates the priority of i for s . Then there is a teacher j such that $\mu_t(j) = s$, $s \succ_i \mu_t(i)$ and $i O_s^t j$. We have two cases: $i >_s^t j$ or $j >_s^t i$. The latter implies that i was originally assigned to school s , that is $i \in \nu_t^{-1}(s)$, but this violates the individual rationality assumption. And the first implies that both i and j were originally assigned to school s , since μ_t is an acceptable matching. But, once again, the latter violates the individual rationality assumption for the assignment of i .

(\Leftarrow) Suppose that μ_t adapts to O^t and is non-wasteful, but not acceptable. Then we have two cases: μ_t is not individually rational or there is a justified claim in μ_t . In the first case, suppose that i is such that $s = \nu_t(i) \succ_i \mu_t(i)$. Since matching μ_t is non-wasteful,

we have teacher j , such that $j \notin \nu_t^{-1}(s)$ and $\mu_t(j) = s$. But then, $i O_s^t j$, and μ_t does not adapt to O^t . If there is a justified claim in μ_t , we have two teachers i, j and a school s , such that $\mu_t(j) = s \succ_i \mu_t(i)$, $i >_s^t j$ and $j \notin \nu_t^{-1}(s)$. But then $i O_s^t j$ and μ_t does not adapt to O^t . ■

Therefore, the problem of finding an acceptable matching in our original framework is equivalent to finding a matching that adapts to O^t and is non-wasteful in the related market.

Proposition 1 *Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, \mathcal{C}_t is not empty.*

Proof. Given the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$, we apply the DA algorithm. It is well-known (see Ergin [8] Proposition 1), that the outcome of the algorithm is a matching that adapts to O^t . It is easy to show that the outcome is also non-wasteful. Then by Lemma 1, we have an acceptable matching for our problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$. ■

Corollary 1 *Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, a fair matching always exists.*

Proof. We know that \mathcal{C}_t is nonempty and finite. For each matching $\mu_t \in \mathcal{C}_t$, compute $|\Gamma(\mu_t)|$. Therefore, we have a finite set of real numbers; take $\mu'_t \in \mathcal{C}_t$ such that $|\Gamma(\mu'_t)| \leq |\Gamma(\mu_t)|$, for all $\mu_t \in \mathcal{C}_t$. Then, μ'_t is a fair matching. ■

We know that in every problem, there is one fair matching. One easily finds examples in which there are more than one fair matching.

It is a classic result of matching theory that the outcome of the DA algorithm satisfies that every agent prefers his partner at this outcome at least as well as the partner of any other acceptable matching. (It is said that the matching is agent-optimal in the subset of acceptable matchings.) Then we know that DA outcome is Pareto superior to any other fair matching. If we proved that the result of the DA algorithm is a fair matching,

we would prove that it is also *the best* fair matching, because it is a well-known result that if preferences are strict, there is only one acceptable matching Pareto superior to any other acceptable matching.⁸ This is the purpose of the following results.

Lemma 2 *Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, consider the outcome of the DA algorithm, denoted by μ_t^{GS} , when it is applied to the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$. Then μ_t^{GS} is a fair matching.*

(See Appendix for a proof).

Since preferences are strict, we have the following characterization theorem.

Proposition 2 *Given an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$, a fair matching μ_t is Pareto superior to any other fair matching if and only if it is the outcome of the DA algorithm (applied to the related market).*

4 A Mechanism

As we have defined, an economy includes a mechanism, because the dynamic of our problem is defined by the relation between the matching of one period and the initial assignment of the following one. We know that the DA mechanism is the *best* one, in the sense that its outcome is fair and Pareto superior to any other fair matching. Then it is the searched mechanism, but it must be applied to the related market defined in Section 3.

Definition 9 *The teacher proposing deferred acceptance mechanism is the mechanism that assigns to each overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ the outcome of the DA algorithm when it is applied to the related market $\langle S, \{q_s\}_s, I^t, \succ, O^t \rangle$.*

⁸See Ergin [8] Proposition 1, and Balinski and Sönmez [3] Theorem 2.

Definition 10 *The teacher proposing deferred acceptance economy is an economy in which the mechanism is the teacher proposing deferred acceptance mechanism.*

Definition 11 *A mechanism is fair if it always selects a fair matching. And an economy is fair if the used mechanism is fair.*

The previous sections show that if we restrict our attention to fair economies, the best economy in terms of efficiency is the teacher proposing deferred acceptance economy. And we also know that essentially it is the unique fair economy with that property (other economies use a mechanism that yields the same result). Hence, we have the following proposition.

Proposition 3 *A mechanism is fair and Pareto superior to any other fair mechanism if and only if it is the teacher proposing deferred acceptance mechanism.*

In the next two sections, we study dynamic properties of the proposed mechanism.

5 Dynamic Strategy-Proofness

Suppose that a new teacher enters the market to compete for a position at time t_0 . A natural question is whether this new teacher can ever benefit by unilaterally misrepresenting her preferences. If the DA mechanism is used, it is a well-known result that she cannot benefit in period t_0 by manipulating her preferences (Roth [11] and Dubins and Freedman [7]). But, what can be said about the following periods? Can a teacher benefit, in the following periods, by sacrificing her school in period t_0 ? After some definitions, we study this issue.

Notation 1 *We denote by $\varphi[I^t, \succ](i)$ the school assigned in period t to teacher i under the mechanism φ .*

Definition 12 Suppose an economy $S, \{q_s\}_s, \nu_1, \{I^t\}_t, \{(\succ_i)_{i \in I^t}\}_t, \{>^t\}_t, \varphi$ and a teacher i who enters the market at time t_0 . We say that the mechanism φ is **dynamic strategy-proof** if teacher i cannot ever benefit by unilaterally misrepresenting her preferences, that is: φ is dynamic strategy-proof if $\varphi[I^t, \succ_{-i}, \succ_i](i) \not\succsim_i \varphi[I^t, \succ_{-i}, \succ'_i](i)$ for all i, \succ_{-i}, \succ'_i and **for all** $t \geq t_0$ such that $i \in I^t$, where \succ_{-i} are the preferences of teachers in the set $I^t \setminus \{i\}$.

Remark 1 The classic concept in static matching problems only makes reference to the benefit in one period. In our framework, the concept involves not only the period when the teacher enters the market (and reveals her preferences) but also all the later periods while she is in the market.

It is interesting to note that a mechanism can be strategy-proof (with the usual static definition) but not dynamic strategy-proof. Appendix A.1 shows a mechanism with this property.

As we remarked in the beginning of this section, when the teacher proposing deferred acceptance mechanism is used in static problems, it is strategy proof. So, we can wonder if this property is also verified by the mechanism in a dynamic context. In the next example, we prove that in our dynamic model, the mechanism can be manipulated by teachers.

Example 2 Consider the following problem:

$$I_E^t = \{j, k\} \subset I^t = \{i, j, k\},$$

$$S = \{s_1, s_2, s_3\}, q_i = 1, i = 1, 2, 3, \nu_t = \{(j, s_2), (k, s_3)\}, \text{ and}$$

the following teacher preferences (from best to worst) and orders:

$$\begin{bmatrix} \succ_i & \succ_j & \succ_k \\ s_2 & s_3 & s_2 \\ s_3 & s_2 & s_3 \\ s_1 & s_1 & s_1 \end{bmatrix} \quad \begin{bmatrix} >_1^t & >_2^t & >_3^t \\ i & j & k \\ j & k & i \\ k & i & j \end{bmatrix}$$

Then the outcome of the teacher proposing deferred acceptance mechanism is:

$$\mu_t = \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \end{pmatrix}$$

For the next period assume: $I^{t+1} = \{i, j, l\}$,

$$\begin{bmatrix} \succ_l \\ s_2 \\ s_3 \\ s_1 \end{bmatrix} \quad \begin{bmatrix} >_1^{t+1} & >_2^{t+1} & >_3^{t+1} \\ i & j & l \\ j & i & i \\ l & l & j \end{bmatrix}$$

The matching in this period is:

$$\mu_{t+1} = \begin{pmatrix} i & j & l \\ s_1 & s_2 & s_3 \end{pmatrix}$$

Suppose that instead of her true preferences, teacher i reveals the following preferences:

$\succ'_i = (s_2, s_1, s_3)$. Then the matching generated in each period is:

$$\mu'_t = \begin{pmatrix} i & j & k \\ s_1 & s_3 & s_2 \end{pmatrix} \quad \mu'_{t+1} = \begin{pmatrix} i & j & l \\ s_2 & s_3 & s_1 \end{pmatrix}$$

Since $\mu'_{t+1}(i) = s_2 \succ_i \mu_{t+1}(i) = s_1$, teacher i can benefit by unilaterally misrepresenting her preferences. ■

Let's examine the last example more closely. By revealing other preferences, teacher i can manipulate the initial submatching of period $t + 1$. When she reveals \succ'_i , teacher j is assigned in period t to school s_3 . Then j has priority over the new teacher l to school s_3 even when she is lower ranked than the new teacher. If i reveals her *true* preferences, new teacher l has priority over j to school s_3 , then j is rejected from that school and she proposes to s_2 , causing the rejection of i from that school. It is easy to see that this case is also possible when there is a unique priority order of all teachers, that is: $>_s^t = >^t$ for all

s and t . However, as we will prove in the next theorem, if in each school's priority ranking teachers that were present in the previous period have priority over new teachers, then the teacher proposing deferred acceptance mechanism is dynamic strategy-proof. First we define this property and then we present our positive result.

Definition 13 *A set of orders $\{>_s^t\}_{s \in S}$ is lexicographic by tenure if for all teachers $i, j \in I^t$, whenever $i \in I_E^t$, and $j \notin I_E^t$ then $i >_s^t j$ for all school $s \in S$.*

In an overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >^t \rangle$ in which the set of orders $>^t = \{>_s^t\}_{s \in S}$ is lexicographic by tenure, each order in the related market consists of three groups of teachers. The first group in the order is the set of teachers who were assigned to the school; then we have the set of teachers who were assigned to another school in the previous period. Finally, we have the new teachers. Within each group, the order is defined by the original priority ranking $>_s^t$.

Definition 14 *An **economy is dynamic strategy-proof** if the used mechanism is dynamic strategy-proof.*

Theorem 2 *Let $S, \{q_s\}_s, \nu_1, \{I^t\}_t, \{(\succ_i)_{i \in I^t}\}_t, \{>^t\}_t, \varphi$ be the teacher proposing deferred acceptance economy. If in each t the set of orders $\{>_s^t\}_{s \in S}$ is lexicographic by tenure, then the economy is dynamic strategy-proof.*

(See Appendix for a proof).

6 Respecting Improvements

In this section, we study another important property of mechanisms, namely, respecting improvements. We say that a mechanism does not respect improvements made by teachers if a teacher may increase her place in one school's priority ranking, everything else remains unchanged, and yet she is punished with a less preferred assignment (Balinski and Sönmez [3]). In the introduction, we presented a mechanism that does not respect

improvements. In this section, we study whether or not the teacher proposing deferred acceptance mechanism has this failure.

Definition 15 *An overlapping teacher placement problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \tilde{\succ}_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$ is an **improvement** for teacher i over another problem $\langle S, \{q_s\}_i, I^t, \mu_{t-1}, \succ, \succ_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$, if $i \succ_{s'}^t j$ implies that $i \tilde{\succ}_{s'}^t j$, and for all teachers k, h different from i , we have that $h \tilde{\succ}_{s'}^t k \Leftrightarrow h \succ_{s'}^t k$.*

According to Definition 15, an improvement for a teacher is basically the original placement problem with the only difference being that the teacher is possibly in a higher place in some school's priority ranking.

Definition 16 *A **mechanism respects improvements** if for any teacher i and $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \tilde{\succ}_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$ an improvement for that teacher over another problem $\langle S, \{q_s\}_i, I^t, \mu_{t-1}, \succ, \succ_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$, the position assigned by the mechanism to teacher i in each period since the improvement (that is, in all periods $\tau \geq t$) is, for teacher i , at least as good as the position assigned in each period beginning with the problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \succ_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$. That is, let μ_t denote the matching that selects the mechanism in the problem with $\succ_{s'}^t$ and $\tilde{\mu}_t$ the matching that selects in the problem with $\tilde{\succ}_{s'}^t$. Then the mechanism **respects improvements** if $\tilde{\mu}_\tau(i) \succeq_i \mu_\tau(i)$ for all $\tau \geq t$.*

Remark 2 *The comment in Remark 1 also applies to this definition. Our concept of respecting improvements involves not only the period when the teacher improves her place in the priority ranking (as in the classic notion), but also every following period while she is in the market.*

It is worth noting that there is no relation between the properties of respecting improvements and dynamic strategy-proofness. Consider the static problem; on the one hand, the mechanism described in the introduction is strategy-proof but does not respect improvements made by teachers (see Appendix A.1). On the other hand, it is straightforward to find a mechanism that respects improvements but is not strategy-proof. Now consider the dynamic problem and a mechanism that is both strategy-proof

and respects improvements (in the static problem). We can wonder if there is any relation between both properties in the dynamic problem. One easily finds examples of mechanisms that satisfy only one of these properties. Hence, there is no relation between these two properties, neither in the static problem nor in the dynamic one.

In the next example, we show that the same problem of the previous section also appears with this property.

Example 3 *Consider the same problem of Example 2 and suppose another problem with the same elements, but in which the order of school s_3 is: $\succ_3^t = (k, j, i)$. Denote by M^t and \bar{M}^t the problem of Example 4 and its modification, respectively. Then, problem M^t represents an improvement for teacher i over \bar{M}^t . The outcome of the teacher proposing deferred acceptance mechanism for each problem is (μ_t corresponds to the problem M^t and $\bar{\mu}_t$ to \bar{M}^t):*

$$\mu_t = \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \end{pmatrix} \quad \bar{\mu}_t = \begin{pmatrix} i & j & l \\ s_1 & s_3 & s_2 \end{pmatrix}$$

In the next period, we have $\succ_3^{t+1} = (l, j, i)$ and the following matchings:

$$\mu_{t+1} = \begin{pmatrix} i & j & k \\ s_1 & s_2 & s_3 \end{pmatrix} \quad \bar{\mu}_{t+1} = \begin{pmatrix} i & j & l \\ s_2 & s_3 & s_1 \end{pmatrix}$$

Finally, we have that $\bar{\mu}_{t+1}(i) \succ_i \mu_{t+1}(i)$. Then, although teacher i improves her position in the ranking of school s_3 , she is assigned in period $t + 1$ to a less preferred school. ■

As we will prove in the next theorem, if the set of orders is lexicographic by tenure, the mechanism respects improvements.

Definition 17 *An **economy respects improvements** if the mechanism used respects improvements.*

Theorem 3 *Consider a teacher i and $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \succ_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$, an improvement for that teacher over another problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, >_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$. Denote by $\tilde{\mu}_t$ and μ_t matchings selected by the teacher proposing*

deferred acceptance mechanism in each problem. Then $\tilde{\mu}_t(i) \succsim_i \mu_t(i)$. Moreover, if in each period $\tau \geq t$ the set of orders is lexicographic by tenure, then the teacher proposing deferred acceptance economy respects improvements.

(See Appendix for a proof).

7 Concluding Remarks

We conclude with a brief discussion about efficiency. A matching μ_t is **Pareto efficient (or simply efficient)** if there is no other matching that makes all teachers present at t weakly better off and at least one teacher strictly better off. A mechanism is efficient if, for any preference profile, it always selects an efficient matching. Then, one can wonder if the proposed mechanism in our model is efficient. We use a result from Ergin [8] to address this question: a cycle for a given priority structure O^t is constituted of distinct schools $s, s' \in S$ and teachers $i, j, k \in I^t$, such that $i O_s^t j O_s^t k O_{s'}^t i$. By Theorem 1 of Ergin [8], we know that the DA mechanism is Pareto efficient if and only if the priority structure is acyclical (that is, the priority structure has no cycle). In our problem, under the assumption that in each period there are at least three teachers, each of whom was assigned to a different school in the previous period, the priority structure of the related market O^t always has at least one cycle. Let $i, j, k \in I_E^t$ with $\nu_t(i) = s$, $\nu_t(j) = s'$ and $\nu_t(k) = s''$, then $i O_s^t j O_s^t k O_{s''}^t i$ or $i O_s^t k O_{s'}^t j O_{s'}^t i$, but in both cases there is a cycle. Finally, applying the mentioned theorem, we know that the proposed mechanism is not Pareto efficient. However, it is important to stress that the outcome of DA mechanism is Pareto efficient in the subset of acceptable matchings. Moreover, since the DA outcome is the unique fair matching that is Pareto superior to any other fair matching, we have the following result: *if in each period there are at least three teachers, each of whom was assigned to a different school in a previous period, there is no fair and efficient mechanism.*

The last result stresses the classic tradeoff between efficiency and fairness (see Abdulkadiroglu and Sönmez [2]). Roughly speaking, one has to choose between one of these

properties. In our model, we consider fairness as more important since once an agent is assigned to a school, she cannot be changed unless she is assigned to a preferred school. In this sense a violation of the fairness condition has consequences in future periods. There are other mechanisms that select Pareto efficient matchings. Gale’s top trading cycles mechanism (described in Abdulkadiroglu and Sonmez [1]) is one of them.

In this paper, we have developed a new framework to model a dynamic school choice problem with overlapping generations of agents. In each period, the central authority must assign teachers to teaching positions. Two elements must be considered in the assignments: the school’s priority rankings and previous assignments. From one period to another, teachers are allowed either to retain their current position, or to choose a preferred one (if available). Hence, the central authority faces a dynamic allocation problem.

The dynamic of our model is defined by the mechanism. The matching in one period links this period with the following one because it determines the initial submatching for the next period. In this framework, we introduced a new fairness concept that is very natural in our context. We have proved that a fair matching always exists and that it can be reached by a modified version of the deferred acceptance algorithm of Gale and Shapley. In particular, the algorithm is applied to a related market in which each school’s priority ranking is modified to obtain an individually rational matching. In relation to the properties of the proposed mechanism, we proved that if the set of orders is lexicographic by tenure, it is dynamic strategy-proof and respects improvements made by teachers.

8 Appendix

A.1 The weaknesses of SEP mechanism.

Suppose there are four schools $S = \{s_1, s_2, s_3, s_4\}$, each one with only one slot and four teachers present in the market at time $t : I^t = \{i, j, k, l\}$. Assume that teachers

k and l were assigned in a previous period to schools s_3 and s_4 , respectively. Teacher preferences (from best to worst) and the ranking are (where \succ_h are preferences of teacher h):

$$\begin{bmatrix} \succ_i & \succ_j & \succ_k & \succ_l \\ s_3 & s_4 & s_1 & s_2 \\ s_1 & s_2 & s_3 & s_3 \\ s_2 & s_3 & s_2 & s_4 \\ s_4 & s_1 & s_4 & s_1 \end{bmatrix} \quad \begin{bmatrix} >^t \\ i \\ k \\ l \\ j \end{bmatrix}$$

That is, teacher i 's most preferred school is s_3 , her second choice is s_1 , and so on. We also have that the first teacher in the ranking is i , the second k , the third l , and the last j . Suppose that we use the mechanism described in the introduction, then the matching in this market is (the school below each teacher is her assigned school):

$$\mu_t = \begin{pmatrix} i & j & k & l \\ s_1 & s_4 & s_3 & s_2 \end{pmatrix}$$

Assume that in the next period, teachers k and l exit the market and two new teachers enter. Then we have $I^{t+1} = \{i, j, m, n\}$, $>^{t+1} = (m, i, n, j)$. The preferences of new teachers are $\succ_m = (s_1, s_3, s_4, s_2)$ and $\succ_n = (s_2, s_4, s_1, s_3)$. Then, the outcome of the mechanism is:

$$\mu_{t+1} = \begin{pmatrix} i & j & m & n \\ s_1 & s_4 & s_3 & s_2 \end{pmatrix}$$

Next we will show how a teacher can benefit by manipulating her preferences. Suppose that instead of her true preferences, teacher i reveals the following preferences: $\succ'_i = (s_3, s_2, s_1, s_4)$. Hence, the outcome of the mechanism in each period is:

$$\mu'_t = \begin{pmatrix} i & j & k & l \\ s_2 & s_4 & s_1 & s_3 \end{pmatrix} \quad \text{and} \quad \mu'_{t+1} = \begin{pmatrix} i & j & m & n \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}$$

Note that $\mu'_{t+1}(i) \succ_i \mu_{t+1}(i)$, and then teacher i benefits in period $t + 1$ by misrepresenting her preferences. Hence, the mechanism is not dynamic strategy-proof. The second flaw we will illustrate is that the mechanism does not respect improvements made by teachers. Suppose that teacher i , instead of being the first in the ranking $>^t$, has a worse

performance and she is the second in the ranking. Specifically, assume that at period t the ranking of teachers is: $\succsim^t = (k, i, l, j)$. Then the outcome of the mechanism is:

$$\tilde{\mu}_t = \begin{pmatrix} i & j & k & l \\ s_3 & s_4 & s_1 & s_2 \end{pmatrix}$$

Therefore, it is better for teacher i to have a lower order in the ranking, because if she increases her position in the ‘priority order,’ like in \succ^t , she will be punished with a worse position. ■

A.2 Proof of Lemma 2

Proof. If $\Gamma(\mu_t^{GS}) = \emptyset$, the proof is complete. Otherwise, we already know that μ_t^{GS} is acceptable. Suppose that it is not fair; then, we have another acceptable matching μ_t , such that $\Gamma(\mu_t) \subsetneq \Gamma(\mu_t^{GS})$. Since μ_t^{GS} is Pareto superior to μ_t : $\mu_t^{GS}(i) \succsim_i \mu_t(i) \forall i$ and there is an agent h such that $\mu_t^{GS}(h) \succ_h \mu_t(h)$. We claim that in this case $\Gamma(\mu_t^{GS}) \subset \Gamma(\mu_t)$, but this contradicts the last relation. Suppose there is a pair $(i, s) \in I^t \times S$, such that $(i, s) \in \Gamma(\mu_t^{GS})$ but $(i, s) \notin \Gamma(\mu_t)$. Then we have a teacher j , such that $\mu_t^{GS}(j) = s \succ_i \mu_t^{GS}(i) = s'$, $i \succ_s^t j$ and $j \in \nu_t^{-1}(s)$. As $(i, s) \notin \Gamma(\mu_t)$, we have two cases: $\mu_t(i) \succsim_i s$ or $j \notin \mu_t^{-1}(s)$. The first case implies $\mu_t(i) \succsim_i s \succ_i \mu_t^{GS}(i)$, but it is not possible since μ_t^{GS} is Pareto superior to all acceptable matchings. In the second case, it must be $\mu_t^{GS}(j) = s = \nu_t(j) \succ_j \mu_t(j)$, but then μ_t is not individually rational. Finally, we prove that $\Gamma(\mu_t^{GS}) \subset \Gamma(\mu_t)$. ■

A.3 Proof of Theorem 5

Proof. For the following proofs, teachers are denoted by i_1, i_2, \dots and schools by s_1, s_2, \dots .

In the **first period**, observe that for strategy-proofness, only agent preferences matter. Then, since in the definition of the related market we do not modify teacher preferences, strategy-proofness in the period when a teacher enters is a direct consequence of Roth [11] and Dubins and Freedman [7]’s results. We shall prove that a teacher cannot benefit by unilaterally misrepresenting her preferences in the following periods while she is in the market.

For the **second period**, suppose teacher i_1 with preferences \succ_{i_1} . We have $\varphi(S, \{q_s\}_s, I^t, \mu_{t-1}, >^t, \succ_{-i_1}, \succ_{i_1})(i_1) = \mu_t(i_1)$, and when i_1 misrepresents her preferences stating \succ'_{i_1} , she obtains $\varphi(S, \{q_s\}_s, I^t, \mu_{t-1}, >^t, \succ_{-i_1}, \succ'_{i_1})(i_1) = \mu'_t(i_1)$. By Roth [11] we know that $\mu_t(i_1) \succsim_{i_1} \mu'_t(i_1)$ and, in particular, if $\mu_t(i_1) \neq \mu'_t(i_1)$ then $\mu_t(i_1) \succ_{i_1} \mu'_t(i_1)$. Each matching at t generates a different initial submatching for the next period. Denote by ν_{t+1} and ν'_{t+1} the initial submatchings define by μ_t and μ'_t , respectively. When mechanism φ is applied to the markets $\langle S, \{q_s\}_s, I^{t+1}, \mu_t, >^{t+1}, \succ_{-i_1}, \succ_{i_1} \rangle$ and $\langle S, \{q_s\}_s, I^{t+1}, \mu'_t, >^{t+1}, \succ_{-i_1}, \succ'_{i_1} \rangle$, matchings μ_{t+1} and μ'_{t+1} are generated. We shall prove that $\mu_{t+1}(i_1) \succsim_{i_1} \mu'_{t+1}(i_1)$.

If $\nu_{t+1} = \nu'_{t+1}$, the argument used in the first period can be applied to prove $\mu_{t+1}(i_1) \succsim_{i_1} \mu'_{t+1}(i_1)$. Then assume $\nu_{t+1} \neq \nu'_{t+1}$. The following lemma will be useful for the proof ■

Lemma 3 *Consider markets $M^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, >^t, \succ_{-i_1}, \succ_{i_1} \rangle$ and $\hat{M}^t = \langle S, \{q_s\}_s, I^t, \mu_{t-1}, >^t, \succ_{-i_1}, \succ'_{i_1} \rangle$ defined before, and the set $I_E^t = \mu_{t-1}^{-1}(S) \cap I^t$. Denote by μ_t and μ'_t the outcome of the teacher proposing deferred acceptance mechanism in each market M^t and \hat{M}^t , respectively. Then every teacher $j \in I_E^t$ satisfies that $\mu'_t(j) = \mu_t(j)$.*

Proof. Suppose, to the contrary, there is a teacher $j_1 \in I_E^t$, such that $\mu'_t(j_1) \neq \mu_t(j_1)$; assume, without loss of generality, $\mu'_t(j_1) \equiv s_1 \succ_{j_1} \mu_t(j_1)$. Then, by non-wastefulness, there is another teacher $j_2 \in I^t$, such that $\mu_t(j_2) = s_1$ and $\mu'_t(j_2) \neq s_1$. Since $\mu_t(j_2) \succ_{j_1} \mu_t(j_1)$, we know that $j_2 O_{s_1}^t j_1$ and in particular $j_2 \in I_E^t$. Therefore, as $\mu'_t(j_2) \neq s_1$, we have $\mu'_t(j_2) \equiv s_2 \succ_{j_2} \mu_t(j_2)$ (otherwise μ'_t does not adapt to O^t). Then, there is another teacher $j_3 \in I^t$, such that $\mu_t(j_3) = s_2$ and $\mu'_t(j_3) \neq s_2$. Since $\mu_t(j_3) \succ_{j_2} \mu_t(j_2)$, we know that $j_3 O_{s_2}^t j_2$ and in particular $j_3 \in I_E^t$. Therefore, as $\mu'_t(j_3) \neq s_2$ we have $\mu'_t(j_3) \equiv s_3 \succ_{j_3} \mu_t(j_3)$ (otherwise μ'_t does not adapt to O^t).

Continuing in this fashion, we construct a cycle (j_1, j_2, \dots, j_n) such that $\mu'_t(j_1) = \mu_t(j_2)$, $\mu'_t(j_2) = \mu_t(j_3)$, ..., $\mu'_t(j_n) = \mu_t(j_1)$ and $\mu'_t(h) \succ_h \mu_t(h)$ for all agents h in the cycle. Next, from matching μ_t , implement the cycle and let μ''_t be this new matching, in which the rest of the teachers keep their position ($\mu''_t(a) = \mu_t(a)$ for all teachers that are not in the cycle). Then, it should be that μ''_t does not adapt to O^t , otherwise μ''_t is a matching

that adapts to O^t and is Pareto superior to μ_t . Then, there is a teacher $k \in I^t$ such that $\mu_t''(j_m) \succ_k \mu_t''(k)$ for some $m = 1, \dots, n$ and $k O_{\mu_t''(j_m)}^t j_m$. Note that $k \in I_E^t$. But since $\mu_t'(j_m) = \mu_t''(j_m)$ and μ_t' adapts to O^t , then $\mu_t'(k) \succ_k \mu_t''(j_m) \succ_k \mu_t''(k) = \mu_t(k)$. Then, we can construct another cycle, implement it, and repeat the procedure until we find a matching that adapts to O^t and is Pareto superior to μ_t ; finally, we have a contradiction.

■

Proof of Theorem 5 (continued).

Proof. Suppose, by way of contradiction, that $\mu'_{t+1}(i_1) \succ_{i_1} \mu_{t+1}(i_1)$ and $\mu'_{t+1}(i_1) = s_1$. Observe that $\nu_{t+1}(i_1) \neq s_1$ and, since $\nu_{t+1}(i_1) \lesssim_{i_1} \nu'_{t+1}(i_1)$, we have $\nu'_{t+1}(i_1) \neq s_1$ (because in the contrary $s_1 \succ_{i_1} \mu_{t+1}(i_1) \lesssim_{i_1} \nu_{t+1}(i_1) \lesssim_{i_1} \nu'_{t+1}(i_1) = s_1$). By non-wastefulness, we know that $|\mu_{t+1}^{-1}(s_1)| = q_{s_1}$ and since $\mu'_{t+1}(i_1) = s_1$, there is a teacher $i_2 \in I^{t+1}$, such that $\mu_{t+1}(i_2) = s_1$ and $\mu'_{t+1}(i_2) \neq s_1$. We claim that $\mu'_{t+1}(i_2) \lesssim_{i_2} s_1$. Suppose $s_1 \succ_{i_2} \mu'_{t+1}(i_2)$. Since $\mu'_{t+1}(i_1) = s_1$, it must be $i_1 \tilde{O}_{s_1}^{t+1} i_2$, and since $s_1 = \mu_{t+1}(i_2) \succ_{i_1} \mu_{t+1}(i_1)$, $i_2 O_{s_1}^{t+1} i_1$ (note that this also implies $i_2 \in I_E^{t+1}$).⁹

We have two cases: $i_1 >_{s_1}^{t+1} i_2$ (and then $\nu_{t+1}(i_2) = s_1$ because $i_1 \in I_E^{t+1}$) or $i_2 >_{s_1}^{t+1} i_1$. In the last case, since $\nu'_{t+1}(i_1) \neq s_1$, we have that $i_2 \notin I_E^t$ (that is, i_2 is a new teacher), which contradicts the fact that $i_2 \in I_E^{t+1}$. In the first case, we claim (1) $\nu'_{t+1}(i_2) \neq s_1$ and (2) $i_2 \in I_E^t$. To prove the first claim, if $\nu'_{t+1}(i_2) = s_1$, we have $s_1 \succ_{i_2} \mu'_{t+1}(i_2) \lesssim_{i_2} \nu'_{t+1}(i_2) = s_1$. For the second claim, we know that $s_1 = \nu_{t+1}(i_2) = \mu_t(i_2) = \mu'_{t+1}(i_1) \succ_{i_1} \mu_{t+1}(i_1) \lesssim_{i_1} \nu_{t+1}(i_1) = \mu_t(i_1)$ and then $i_2 O_{s_1}^t i_1$. But as $i_1 >_{s_1}^t i_2$, then it must be $i_2 \in I_E^t$. Finally, we found that $i_2 \in I_E^t$ and $s_1 = \nu_{t+1}(i_2) = \mu_t(i_2) \neq \nu'_{t+1}(i_2) = \mu'_t(i_2)$. But, due to the last lemma, this is a contradiction. Then $\mu'_{t+1}(i_2) \lesssim_{i_2} \mu_{t+1}(i_2) = s_1$ and because of $\mu'_{t+1}(i_2) \neq s_1$, we have $\mu'_{t+1}(i_2) \equiv s_2 \succ_{i_2} \mu_{t+1}(i_2) = s_1$ and $i_2 \in I_E^{t+1}$.

For i_2 we have a similar situation as for i_1 : i_2 is proposed to s_2 in the problem with ν_{t+1} and she is rejected. By non-wastefulness, we know that $|\mu_{t+1}^{-1}(s_2)| = q_{s_2}$ and due to $\mu'_{t+1}(i_2) = s_2$, there is a teacher i_3 such that $\mu_{t+1}(i_3) = s_2$ and $\mu'_{t+1}(i_3) \neq s_2$. As before, we

⁹Here $\tilde{O}_{s_1}^{t+1}$ denotes the strict priority order in the related market of all teachers for school s_1 in the problem with ν'_{t+1} .

claim that $\mu'_{t+1}(i_3) \succsim_{i_3} s_2$. Suppose, to the contrary, $s_2 \succ_{i_3} \mu'_{t+1}(i_3)$. Since $\mu'_{t+1}(i_2) = s_2$, it must be $i_2 \tilde{O}_{s_2}^{t+1} i_3$ and, since $\mu_{t+1}(i_3) \succ_{i_2} \mu_{t+1}(i_2)$, $i_3 O_{s_2}^{t+1} i_2$ (notice that this also implies $i_3 \in I_E^{t+1}$). We have two cases to study: $i_2 >_{s_2}^{t+1} i_3$ (and then $\nu_{t+1}(i_3) = s_2$ because $i_2 \in I_E^{t+1}$) or $i_3 >_{s_2}^{t+1} i_2$ (and then $\nu'_{t+1}(i_2) = s_2$).

In the first case, we claim: (1) $\nu'_{t+1}(i_3) \neq s_2$ and (2) $i_3 \in I_E^t$. If $\nu'_{t+1}(i_3) = s_2$, we have $s_2 \succ_{i_3} \mu'_{t+1}(i_3) \succsim_{i_3} \nu'_{t+1}(i_3) = s_2$. We also know that $s_2 = \nu_{t+1}(i_3) = \mu_t(i_3) = \mu'_{t+1}(i_2) \succ_{i_2} \mu_{t+1}(i_2) \succsim_{i_2} \nu_{t+1}(i_2) = \mu_t(i_2)$ and then $i_3 O_{s_2}^t i_2$. But as $i_2 >_{s_2}^t i_3$, then it must be $i_3 \in I_E^t$. Finally, we found that $i_3 \in I_E^t$ and $s_2 = \nu_{t+1}(i_3) = \mu_t(i_3) \neq \nu'_{t+1}(i_3) = \mu'_t(i_3)$. But, because of the last lemma, this is a contradiction.

In the second case, we claim (1) $\nu_{t+1}(i_2) \neq s_2$ and (2) $i_2 \in I_E^t$. To prove the first claim, note that if $\nu_{t+1}(i_2) = s_2$, we have $s_2 \succ_{i_2} \mu_{t+1}(i_2) \succsim_{i_2} \nu_{t+1}(i_2) = s_2$. For the second claim, we know that $s_2 = \mu'_t(i_2) = \mu_{t+1}(i_3) \succ_{i_3} \mu'_{t+1}(i_3) \succsim_{i_1} \nu'_{t+1}(i_3) = \mu'_t(i_3)$ and then $i_2 O_{s_2}^t i_3$. But as $i_3 >_{s_2}^t i_2$, then it must be $i_2 \in I_E^t$. Observe that $i_2 \in I_E^t$ and $s_2 = \nu'_{t+1}(i_2) = \mu'_t(i_2) \neq \nu_{t+1}(i_2) = \mu_t(i_2)$, which is a contradiction to the last lemma.

Thus, $\mu'_{t+1}(i_3) \succsim_{i_3} \mu_{t+1}(i_3) = s_2$ and due to $\mu'_{t+1}(i_3) \neq s_2$, we have $\mu'_{t+1}(i_3) \equiv s_3 \succ_{i_3} \mu_{t+1}(i_3) = s_2$ and $i_3 \in I_E^{t+1}$. Continuing in this fashion, we construct a cycle (i_1, i_2, \dots, i_n) such that $\mu'_{t+1}(i_1) = \mu_{t+1}(i_2)$, $\mu'_{t+1}(i_2) = \mu_{t+1}(i_3)$, ..., $\mu'_{t+1}(i_n) = \mu_{t+1}(i_1)$ and $\mu'_{t+1}(h) \succ_h \mu_{t+1}(h)$ for all agents in the cycle. Next, from matching μ_{t+1} we implement the cycle and μ''_{t+1} is this new matching in which the rest of teachers keep their position ($\mu''_{t+1}(a) = \mu_{t+1}(a)$ for all teachers that are not in the cycle). Then, it must be that μ''_{t+1} does not adapt to O^{t+1} , otherwise μ''_{t+1} is a matching that adapts to O^{t+1} and is Pareto superior to μ_{t+1} . Then, there is a teacher $k \in I^{t+1}$, such that $s_m \equiv \mu''_{t+1}(i_m) \succ_k \mu''_{t+1}(k)$ for some $m = 1, \dots, n$ and $k O_{s_m}^{t+1} i_m$. Note that $k \in I_E^{t+1}$. But since $\mu'_{t+1}(i_m) = \mu''_{t+1}(i_m)$ and μ'_{t+1} adapts to \tilde{O}^t , we have two cases: $\mu'_{t+1}(k) \succ_k \mu''_{t+1}(i_m) \succ_k \mu''_{t+1}(k) = \mu_{t+1}(k)$ or $\mu'_{t+1}(i_m) = \mu''_{t+1}(i_m) \succ_k \mu'_{t+1}(k)$ and $i_m \tilde{O}_{s_m}^{t+1} k$.

In the first case, we have that k has a better position in μ'_{t+1} than in μ_{t+1} , and then we can construct another cycle. We claim that the second case is not possible. Once again, we analyze two cases: $i_m >_{s_m}^{t+1} k$ and $k >_{s_m}^{t+1} i_m$. If $i_m >_{s_m}^{t+1} k$, as $i_m, k \in I_E^{t+1}$

and $k \succ_{s_m}^{t+1} i_m$, then $\nu_{t+1}(k) = s_m$; we claim that: (1) $\nu'_{t+1}(k) \neq s_m$ and (2) $k \in I_E^t$. First, observe that if $\nu'_{t+1}(k) = s_m$, we have $s_m \succ_k \mu'_{t+1}(k) \succsim_k \nu'_{t+1}(k) = s_m$. We also know that $s_m = \nu_{t+1}(k) = \mu_t(k) = \mu'_{t+1}(i_m) \succsim_{i_m} \nu'_{t+1}(i_m) = \mu'_t(i_m)$ and then $k \succ_{s_m}^t i_m$. But as $i_m \succ_{s_m}^t k$, then it must be $k \in I_E^t$. Finally we found that: $k \in I_E^t$ and $s_m = \nu_{t+1}(k) = \mu_t(k) \neq \nu'_{t+1}(k) = \mu'_t(k)$. But, because of the last lemma, this is a contradiction. A similar argument shows in the second case that $\nu'_{t+1}(i_m) = s_m \neq \nu_{t+1}(i_m)$ and $i_m \in I_E^t$, and then we find another contradiction. Finally, as we note earlier, in the first case we can construct another cycle and repeat the procedure until we find a matching that adapts to O^t and is Pareto superior to μ_t ; then we have a final contradiction.

For the next period, we have that $\mu_{t+1}(i_1) \succsim_{i_1} \mu'_{t+1}(i_1)$ and the same argument applies to prove that i_1 can never benefit in the following periods. ■

A.4 Proof of Theorem 6

Proof. For the **first period**, we have problem $\langle S, \{q_s\}_i, I^t, \mu_{t-1}, \succ, \succ_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$ and another problem $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \succ_{s'}^t, \{\succ_s^t\}_{s \neq s'} \rangle$, which represents an improvement for a teacher i_1 . Let μ_t denote the matching selected by the teacher proposing deferred acceptance mechanism in the first problem and $\tilde{\mu}_t$ the one selected in the second problem. We shall prove that $\tilde{\mu}_t(i_1) \succsim_{i_1} \mu_t(i_1)$. Suppose that $\mu_t(i_1) \succ_{i_1} \tilde{\mu}_t(i_1)$ and let $\mu_t(i_1) \equiv s_1$. In the related market of each problem, we know that $O_s^t = \tilde{O}_s^t$ for all $s \neq s'$, $j \succ_{s'}^t k \iff j \tilde{\succ}_{s'}^t k$, if $i_1 \succ_{s'}^t k$ then $i_1 \tilde{\succ}_{s'}^t k$ and if $j \tilde{\succ}_{s'}^t i_1$ then $j \succ_{s'}^t i_1$.¹⁰

Then, as $\tilde{\mu}_t$ is non-wasteful, we have $|\tilde{\mu}_t^{-1}(s_1)| = q_{s_1}$ and then, there is a teacher $i_2 \in I^t$ such that $\tilde{\mu}_t(i_2) = s_1$ and $\mu_t(i_2) \neq s_1$. Since $\tilde{\mu}_t(i_2) \succ_{i_1} \tilde{\mu}_t(i_1)$, we know that $i_2 \tilde{\succ}_{s_1}^t i_1$ and then $i_2 \succ_{s_1}^t i_1$. Therefore, as $\mu_t(i_1) = s_1$, then $\mu_t(i_2) \succ_{i_2} s_1 = \tilde{\mu}_t(i_2)$. Let $\mu_t(i_2) \equiv s_2$. We can make the same argument for i_2 . As $\tilde{\mu}_t$ is non-wasteful, there is a teacher $i_3 \in I^t$ such that $\tilde{\mu}_t(i_3) = s_2$ and $\mu_t(i_3) \neq s_2$. Since $\tilde{\mu}_t(i_3) \succ_{i_2} \tilde{\mu}_t(i_2)$, we know that $i_3 \tilde{\succ}_{s_2}^t i_2$ and then $i_3 \succ_{s_2}^t i_2$. Therefore, as $\mu_t(i_2) = s_2$, then $\mu_t(i_3) \succ_{i_3} s_2 = \tilde{\mu}_t(i_3)$.

¹⁰Here $\tilde{O}_{s'}^{t+1}$ denotes the strict priority order in the related market of all teachers for school s' in the problem with $\tilde{\succ}_{s'}^t$.

Continuing in this fashion, we construct a cycle (i_1, i_2, \dots, i_n) such that $\mu_t(i_1) = \tilde{\mu}_t(i_2)$, $\mu_t(i_2) = \tilde{\mu}_t(i_3)$, ..., $\mu_t(i_n) = \tilde{\mu}_t(i_1)$ and $\mu_t(h) \succ_h \tilde{\mu}_t(h)$ for all agents in the cycle. Next, from matching $\tilde{\mu}_t$ implement the cycle and μ'_t is this new matching; that is $\mu'_t(i) = \mu_t(i)$ for all $i \in \{i_1, i_2, \dots, i_n\}$ and $\mu'_t(i) = \tilde{\mu}_t(i)$ for all $i \notin \{i_1, i_2, \dots, i_n\}$. Then, it must be that μ'_t does not adapt to \tilde{O}^t ; otherwise μ'_t is a matching that adapts to \tilde{O}^t and is Pareto superior to $\tilde{\mu}_t$. Then, there is a teacher $j \notin \{i_1, i_2, \dots, i_n\}$ such that $s_m \equiv \mu'_t(i_m) \succ_j \mu'_t(j)$ for some $m = 1, \dots, n$ and $j \tilde{O}_{s_m}^t i_m$. But since $\mu'_t(i_m) = \mu_t(i_m)$ and μ_t adapts to O^t , we have two cases: $\mu_t(j) \succ_j \mu'_t(i_m) \succ_j \mu'_t(j)$ or $\mu_t(i_m) = \mu'_t(i_m) \succ_j \mu_t(j)$ and $i_m O_{s_m}^t j$. In the second case, it must be that $s_m = s'$ and $j = i_1$, and then we have a contradiction. Hence, it is the case that $\mu_t(j) \succ_j \mu'_t(j)$ and we can construct another cycle and repeat the procedure until we find a matching that adapts to \tilde{O}^t and is Pareto superior to $\tilde{\mu}_t$; then we have a final contradiction.

For the **second period**, denote by $M^{t+1} = \langle S, \{q_s\}_s, I^{t+1}, \mu_t, \succ, >_{s'}^{t+1}, \{>_s^{t+1}\}_{s \neq s'} \rangle$ and $\tilde{M}^{t+1} = \langle S, \{q_s\}_s, I^{t+1}, \tilde{\mu}_t, \succ, \tilde{>}_{s'}^{t+1}, \{\tilde{>}_s^{t+1}\}_{s \neq s'} \rangle$ the markets of the following period in the case that we have in period t : $\langle S, \{q_s\}_i, I^t, \mu_{t-1}, \succ, >_{s'}^t, \{>_s^t\}_{s \neq s'} \rangle$ and $\langle S, \{q_s\}_s, I^t, \mu_{t-1}, \succ, \tilde{>}_{s'}^t, \{\tilde{>}_s^t\}_{s \neq s'} \rangle$, respectively.; μ_{t+1} and $\tilde{\mu}_{t+1}$ denote the outcome of the mechanism in each market in period $t + 1$. We define the following market at $t + 1$: $\hat{M}^{t+1} = \langle S, \{q_s\}_s, I^{t+1}, \mu_t, \succ, \tilde{>}_{s'}^{t+1}, \{\tilde{>}_s^{t+1}\}_{s \neq s'} \rangle$ and $\hat{\mu}_{t+1}$ is the matching reached by the mechanism in this market. We know that $\tilde{\nu}_{t+1}(i_1) \tilde{\succ}_{i_1} \nu_{t+1}(i_1) = \hat{\nu}_{t+1}(i_1)$.¹¹ Then by the argument used in strategy-proofness, we have $\tilde{\mu}_{t+1}(i_1) \tilde{\succ}_{i_1} \hat{\mu}_{t+1}(i_1)$, and by respecting improvement for one period (the statement just proved), $\hat{\mu}_{t+1}(i_1) \tilde{\succ}_{i_1} \mu_{t+1}(i_1)$. Finally, by transitivity, we have $\tilde{\mu}_{t+1}(i_1) \tilde{\succ}_{i_1} \mu_{t+1}(i_1)$. Repeating the argument for the following periods, we have $\tilde{\mu}_\tau(i_1) \tilde{\succ}_{i_1} \mu_\tau(i_1)$ for all $\tau \geq t$. ■

References

- [1] A. Abdulkadiroglu, T. Sönmez, House allocation with existing tenants, Journal of Economic Theory 88 (1999) 233-260.

¹¹ $\hat{\nu}_{t+1}$ denotes de initial submatching of the market \hat{M}^{t+1} .

- [2] A. Abdulkadiroglu, T. Sönmez, School choice: A mechanism design approach, *The American Economic Review* 93 (3) (2003) 729-747.
- [3] M. Balinski, T. Sönmez, A tale of two mechanisms: Student placement, *Journal of Economic Theory* 84 (1999) 73-94.
- [4] F. Bloch, D. Cantala, Markovian assignment rules, working paper (2008).
- [5] D. Cantala, Eficiencia y justicia en la administración publica: la educación en México, mimeo El Colegio de México (2008).
- [6] O. Compte, P. Jehiel, Voluntary participation and re-assignment in two-sided matching, working paper (2008).
- [7] L. Dubins, D. Freedman, Machiavelli and the Gale Shapley algorithm, *American Mathematical Monthly* 88 (1981) 485-494.
- [8] H. Ergin, Efficient resources allocation on the basis of priorities, *Econometrica* 70 (6) (2002) 2489-2497.
- [9] D. Gale, L. Shapley, College admissions and the stability of marriage, *American Mathematical Monthly* 69 (1962) 9-15.
- [10] M. Kurino, House allocation with overlapping agents: A dynamic mechanism approach, *Jena Economic Research Papers* 2009-075 (2009).
- [11] E. Roth, The economics of matching: Stability and incentives, *Mathematics of Operations Research* 7 (1982) 617-628.
- [12] E. Roth, M. Sotomayor, Two-sided matching: A study in game-theoretic modeling and analysis, Cambridge, Cambridge University Press (1990).
- [13] T. Sönmez, U. Ünver, Matching, allocation, and exchange of discrete resources, Jess Benhabib, Alberto Bisin, and Matthew Jackson (eds.), *Handbook of Social Economics*, Elsevier, (2009).

- [14] U. Ünver, Dynamic kidney exchange, *Review of Economic Studies* 77 (1) (2010) 372-414.